Simulation of the Hard Core by a Velocity Dependence*

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Recently, Green has shown explicitly that different potentials which fit the same S-wave phase shifts from 0 to 250 MeV (lab system) can give quite different binding energies of nuclear matter. In this paper we show, for two particular potentials, a velocity-dependent one vs one with a hard core, how this different behavior is related to the form of the two-particle wave function at small interparticle distances. The equality of the phase shift vs energy for two potentials requires that the two-particle wave function must have not only the same asymptotic form but also the same value of an effective-range integral, which places some constraint on the form of the interior wave function as well. It is shown that the potentials give also nearly the same results to first order in the "separation method," an approximation to the treatment of nuclear matter in which the wave function is assumed to equal the interior wave function at short distances joining on smoothly with the unperturbed wave function at a distance of about 1 F. However, the change in the particle propagator due to the velocity dependence of the nucleon-nucleus potential gives a positive contribution to the energy, which depends on the wave function at short distances. This contribution is much bigger for the hard-core potential than for the velocity-dependent one, since there is more of a short-range correlation effect in the former case.

I. INTRODUCTION

It was suggested by Peierls[1] that the hard core in the nucleon-nucleon interaction might be replaced by a less singular, but strongly velocity-dependent repulsion. Recently, several authors have investigated this problem in some detail.[2–4] By suitably adjusting parameters, it was possible to give almost as good fits to observed nucleon-nucleon scattering cross sections with a velocity dependence as with a hard core. Indeed, it has been shown explicitly by Baker[5] that the scattering phase shifts vs energy resulting from a potential containing a velocity-dependent repulsion are *exactly* the same as those for an angular momentum dependent potential outside a hard core. Since the velocity-dependent potential is nonsingular, the binding energy of nuclear matter can be calculated using a perturbation expansion. This was done by Green,[3] who found that (notwithstanding some uncertainty in the magnitude of the higher order terms) such a potential would give several MeV per particle too much binding of nuclear matter. Conversely, Brueckner and Masterson[6] have shown that the Breit potential,[7] which has a somewhat larger core radius than the Gammel-Thaler potential[8] but gives nearly the same phase shifts, leads to a considerably smaller binding of nuclear matter (8 vs 16 MeV/A). Green then investigated this problem more closely by taking a velocity-dependent potential[9] whose parameters are adjusted to give practically the same S-wave phase shifts from 0 to 250 MeV (lab energy) as a core plus exponential well potential considered previously by Scott and the present author.[10] The latter potential has its parameters adjusted to give a bound state at 0 energy, an effective range 2.5 F, and a core radius of 0.4 F. Its S-wave phase shift is close to the empirical ${}^1S_0$ phase shift and it becomes negative at about 230-MeV lab energy. If both of these potentials are arbitrarily assumed to act only in S states of relative motion, then they, of course, give the same scattering cross sections up to at least 250 MeV. (Actually Green considered several different velocity-dependent potentials, all of which give nearly the same phase shifts, but his potential No. 3 fits ours particularly accurately, and will be the only one considered in this paper.) Of course, as is well known,[11,12] two potentials which fit the same S-wave phase shift vs energy (and bound-state energy, if any) must be identical if it is required that (1) both potentials be static, i.e., no velocity dependence such as is considered in the present paper and (2) the fit extends to *all* energies, not just up to 250 MeV. In MS it was shown that the core+exponential potential would lead to an energy minimum of about 10 MeV per particle at a Fermi momentum k_F of 1.4 F^{-1} corresponding to a radius $R_0=1.08A^{1/3}$ F. However, Green[9] finds for his velocity-dependent potential an energy 18 MeV/Å (using second order perturbation theory and a modified propagator) at this value of k_F and no energy minimum at all (for $R_0>0.5A^{1/3}$ F). Thus, apparently, two potentials fitting the same scattering data (since they are assumed to

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[1] R. E. Peierls, in *Proceedings of the International Conference on Nuclear Structure, Kingston*, edited by D. A. Bromley and E. Vogt (University of Toronto Press, Toronto, 1960), p. 7.
[2] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962); O. Rojo and L. M. Simmons, *ibid* **125**, 273 (1962).
[3] A. M. Green, Nucl. Phys. **33**, 218 (1962).
[4] C. Dismukes, Ph.D. dissertation, University of California, Los Angeles, 1962 (unpublished).
[5] G. A. Baker, Jr., Phys. Rev. **128**, 1485 (1962).
[6] K. A. Brueckner and K. S. Masterson, Phys. Rev. **128**, 2267 (1962).
[7] G. Breit, M. H. Hull, K. E. Lassila, and K. D. Pyatt, Phys. Rev. **120**, 2227 (1960). M. H. Hull, K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, *ibid.* **122**, 1606 (1961).
[8] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291, 1337 (1957).
[9] A. M. Green, Phys. Letters **1**, 136 (1962).
[10] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960) referred to in the present paper as MS.
[11] I. M. Gelfand and B. M. Levitan, Doklady Akad. Nauk S.S.S.R. **77**, 557 (1951).
[12] R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952).

act only in S states) nevertheless can give quite different results for nuclear matter.

In this paper we would like to point out that this result follows directly from the different behavior of the two-particle wave function at short distances.

II. EFFECTIVE RANGE THEORY

Figure 1 shows the two S-state wave functions at three different energies.[13] The wave functions are normalized so that at large distances:

$$R(k,r) \underset{r\to\infty}{\longrightarrow} S(k,r) = \sin[kr + \delta_S(k)]. \quad (1)$$

The values $k = 0.7$ and $1.4\ \mathrm{F}^{-1}$ correspond approximately to the average and maximum relative momentum of a nucleon pair in nuclear matter (for $k_F = 1.4\ \mathrm{F}^{-1}$). It is seen that the two wave functions have practically the same asymptotic form (i.e., same phase shift) at all energies but the wave function R_c for the core+exponential potential (V_c) has much more of a "wound" in it than the wave function R_v for the velocity-dependent potential (V_v). Now the fact that the two potentials give the same S-wave phase shifts δ_S at all momenta implies that they also give the same derivative $d\delta_S/dk$. Using effective range theory this derivative can be expressed in terms of the wave function at the given momentum. By a straightforward calculation it can be shown that

$$\frac{d\delta_S}{dk} = \frac{\sin 2\delta_S(k)}{2k} - 2\int_0^\infty [S^2(k,r) - R^2(k,r)]dr, \quad (2)$$

where $S(k,r)$ is the asymptotic wave function as defined in Eq. (1). Thus for two potentials giving the same phase shift at all energies, the wave functions do not only have the same asymptotic form, but also the same value of the integral $\int_0^\infty R^2 dr$. This is, indeed, satisfied as can be seen explicitly in Fig. 2 for the case $k = 0.7\ \mathrm{F}^{-1}$. The difference between the wave functions is much less noticeable in a plot of R^2 than one of R itself. The two shaded regions where the wave functions do not overlap have nearly the same small area, so that their contributions to the integral $\int R^2 dr$ essentially cancel. Thus, while the equlity of the phase shifts implies some condition on the interior wave function, it does not seriously restrict the form of the wave function at short distances.

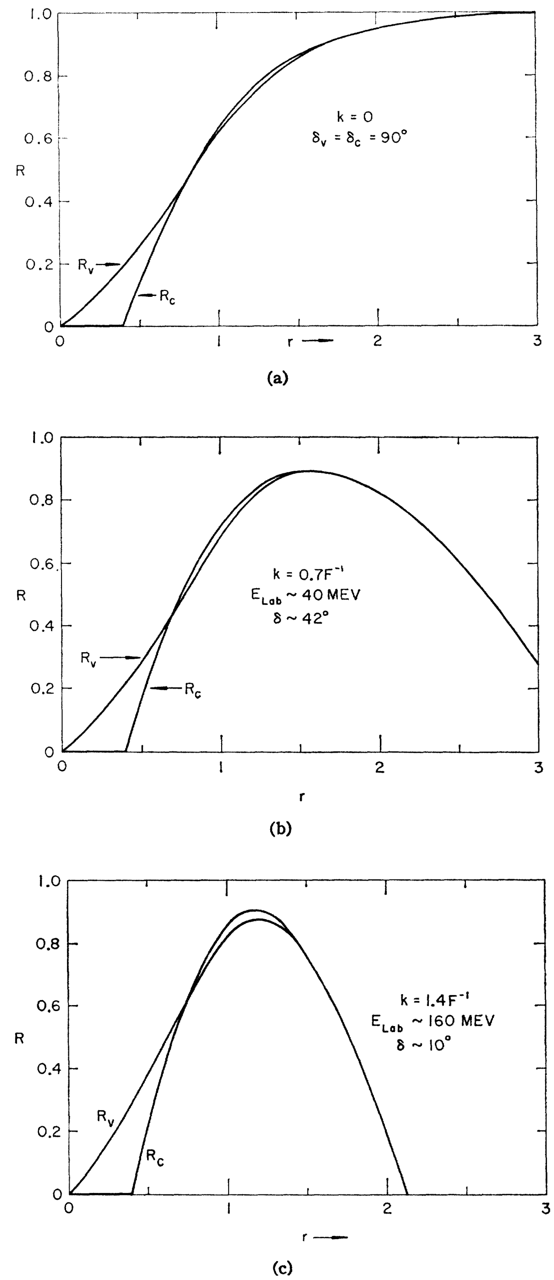

FIG. 1. Comparison of two-particle S-state wave functions for velocity-dependent potential (Potential No. 3 of reference 9) and core+exponential potential (reference 10), which give nearly the same S-wave phase shift from 0 to 250 MeV (lab energy). (a), (b), and (c) show the respective wave functions R_v and R_c vs interparticle distance for relative momenta $k=0$, $0.7\ \mathrm{F}^{-1}$ and $1.4\ \mathrm{F}^{-1}$.

III. NUCLEAR MATTER—FIRST-ORDER SEPARATION METHOD

A quite accurate idea for the effect of nuclear matter on the two-particle wave function can be obtained by dividing the wave function space into two regions as suggested in MS. At the separation point $r = d(k)$ the logarithmic derivative (rR'/R) is equal to the value $kd \cot kd$ for the unperturbed wave function. It was shown in MS that the two-particle wave function in nuclear matter is quite well approximated by a wave function, called R^s, which coincides with the interacting two-particle wave function for $r \leq d$, but equals the unperturbed wave function at larger distances. In Fig. 3, we plot R^s for the two potentials at $k = 0.7\ \mathrm{F}^{-1}$. The

[13] A. M. Green (private communication).

normalization is now chosen differently from before:

$$R^s(k,r) \rightarrow S(k,r) = \sin kr/k. \tag{3}$$

The first-order contribution to the potential energy is given by[10]

$$PE^{(1)} = \frac{k_F^3}{4\pi^2}\langle V^L(k,k)\rangle_{\rm av}, \tag{4}$$

where

$$V^L(k',k) = 4\pi\int_0^\infty S(k',r)V(\nabla,r)R^S(k,r)dr. \tag{5}$$

As a rough guide, we may set[10]

$$\langle V^L(k,k)\rangle_{\rm av} = V^L(0.7,0.7). \tag{6}$$

Incidentally, for a velocity-dependent potential of the form considered here:

$$V(\nabla,r) = U(r) - [\nabla^2\cdot W(r) + W(r)\cdot\nabla^2], \tag{7}$$

it is easy to show that

$$V^L(k,k) = 4\pi\int_{d(k)}^\infty (U+2k^2W)S^2(k,r)dr + 4\pi W'(d)S^2(k,d). \tag{8}$$

By numerical integration we find for the velocity-dependent potential:

$$V_v^L(0.7,0.7) = -553 \text{ MeV F}^3. \tag{9}$$

This result is surprisingly close to that for a core+exponential potential. For the latter, we obtain

$$V_c^L(k,k) = 4\pi\int_0^\infty S(k,r)V(r)R_c^S(k,r)dr = 4\pi\int_{d(k)}^\infty V(r)S^2(k,r)dr, \tag{10}$$

and as was shown in MS

$$V_c^L(0.7,0.7) = -542 \text{ MeV F}^3. \tag{11}$$

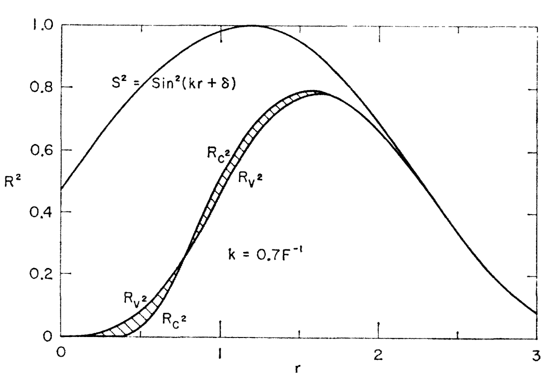

FIG. 2. Comparison of R_v^2 and R_c^2 with asymptotic form.

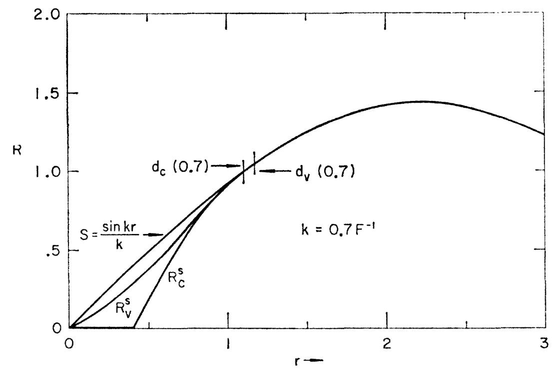

FIG. 3. Comparison of wave functions R_v^s and R_c^s which coincide with interacting two-particle wave function for $r \leq d(k)$ and with the unperturbed wave function $S = \sin kr/k$ for $r \geq d(k)$. Also indicated are the separation distances for the two wave functions.

Since 1 MeV F^3 corresponds to 0.069 MeV/Å binding at $k_F^{-1} = 1.4$ F^{-1}, the first-order binding energy for the two potentials differs by less than an MeV/Å. If we use, instead, conventional first-order perturbation theory, we obtain

$$V_v^1(k',k) = 4\pi\int_0^\infty S(k',r)V(\nabla,r)S(k,r)dr = 4\pi\int_0^\infty [U+(k^2+k'^2)W]S(k',r)S(k,r)dr. \tag{12}$$

For a hard-core potential, all matrix elements are infinite, while for the velocity-dependent potential, we find

$$V_v^1(0.7,0.7) = -493 \text{ MeV F}^3. \tag{13}$$

The short-range correlation correction

$$\Delta V = V^L - V^1 = 4\pi\int_0^\infty SV(R-S)dr. \tag{14}$$

evidently contributes $-\infty$ for the core+exponential potential and about -4 MeV/Å for the velocity-dependent potential. Indeed, while conventional perturbation theory is *obviously* inapplicable for a hard core potential, it also does not converge well for the velocity-dependent case either. The far-off diagonal elements $k \sim 0.7$ F^{-1}, $k' \gg 1.4$ F^{-1} of the repulsive term are quite large due to the presence of the factor $(k')^2$ on the right-hand side of Eq. 12; and Green[9] obtains a second-order energy of -10 MeV/Å even with a modified propagator.

Table I summarizes the results for $k = 0$, 0.7, and 1.4 F^{-1} and it is seen that at least to first order in the separation method, the two potentials considered give nearly the same results.

So far, our results bear out Bég's argument[14] that potentials which are equivalent in the two-body problem

[14] M. Bég, Ann. Phys. (N. Y.) 13, 110 (1961).

TABLE I. Comparison of first-order energies for velocity-dependent V_v and core+exponential V_c potentials. $V^1(k,k)$ is the first-order result in conventional perturbation theory. $V^L(k,k)$ is the first-order result in the separation method. All matrix elements are expressed in units MeV F.[3]

k	V_v^1	V_v^L	V_c^1	V_c^L
0	-1000	-1052	$+\infty$	-1031
0.7	-493	-553	$+\infty$	-543
1.4	$+21$	-78	$+\infty$	-76

will also give nearly the same results for nuclear matter at not too high densities. The wave functions R^s are close to the two-particle eigenfunctions for an intermediate density. The effect of the other nucleons is assumed to be large enough to wipe out the long-range correlations (which would, for example, give a phase shift) and yet too small to have any effect on the short-range correlations associated with the soft or hard core. It appears from our results that Bég's argument does, indeed, apply for densities from 0 up to values at which the effect of modified particle propagation on the short-range correlations becomes important. Unfortunately, as we shall see presently, the latter appears to be already the case at normal nuclear density.

IV. NUCLEAR MATTER—THE DISPERSION TERM

Up to now, we have neglected any change in the particle propagator due to the presence of the other particles (except, of course, for the Pauli principle which suppresses long-range correlations).

However, due to the well-known momentum dependence of the nucleon-nucleus potential, there will be a significant difference in the binding energy of nuclear matter for our two potentials. A very crude way of representing this effect is the so-called "effective-mass" approximation, which assumes that the potential varies quadratically with momentum. A considerably better approximation is to set the one-body potential equal to a constant at high momenta. This gives the so-called "dispersion approximation"[10,15] or "reference spectrum."[16] The dominant second-order correction term to the separation method is the so-called dispersion term given by

$$V^D = 8\pi\Delta U \int_0^d (S-R^S)^2 dr, \qquad (15)$$

where ΔU is the extra one-body potential for a particle excited far out of the Fermi sea compared to one in the Fermi sea. Taking $\Delta U = 60$ MeV,[10] we obtain for the dispersion contribution 5 MeV/Å for the core potential, but only 0.6 MeV/Å for the velocity-dependent case. This seems to the major cause of the larger binding energy in the latter case and especially, for the absence of saturation, since the dispersion term turns out to be strongly density dependent. Rough estimates indicate that the other second-order terms (in the sense of MS) contribute less than 1 MeV/Å for the velocity-dependent potential so that the separation method appears to converge more rapidly than for the core+exponential potential.

[15] H. S. Köhler, Ann. Phys. (N. Y.) **16**, 375 (1961).
[16] H. A. Bethe, B. H. Brandow, and A. Petschek, Phys. Rev. **129**, 225 (1963).

V. CONCLUSIONS

As we have seen, the behavior of the two-particle wave function at short distances has only a very minor effect on the dependence of S-wave phase shift vs energy, but an important one on the interaction of two particles in nuclear matter.

Thus, it appears that elastic scattering data alone even up to 300 MeV may not be sufficient to specify the nucleon-nucleon potential accurately enough for purposes of many-body calculations. Other information seems to be required in order to deduce the behavior of the wave function at short interparticle distances.

Conceivably, it might even prove helpful to use binding energies of complex nuclei to specify the nucleon-nucleon interaction more precisely.

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